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On Prime Jordan Rings $H(R)$ with Chain Condition

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In [3], Jacobson gives conditions in Jordan theory analogous to the conditions of Ore's theorem for associative integral domains. He suggests that these conditions on a Jordan ring J might imply that J can be embedded in a Jordan division ring.

Here we restrict our attention to the Jordan ring of symmetric elements of an associative ring with involution. Although we consider the problem of integral domains in this restricted case our main result is more general.

Our approach is via Goldie's theorem [2] for associative rings: T has a ring of quotients which is semi-simple Artinian if and only if T is semi-prime, contains no infinite direct sum of left ideals and satisfies ACC on left annihilator ideals. If one replaces semi-prime by prime, then we replace semi-simple by simple. One can easily see that the conditions put on left ideals is implied by ACC or DCC on left ideals, when T has an involution. From this point of view we state our main result.

MAIN THEOREM. *If R is an associative ring with involution $*$ of characristic $\neq 2$ such that the Jordan ring of symmetric elements, H, is prime and satisfies either ACC or DCC on quadratic ideals and if M is the maximal $*$-ideal in R such that $M \cap H = 0$ then R/M has a ring of quotients which is $*$-simple Artinian.*

We shall see that the main theorem implies that H has a "Jordan ring of quotients" which is of the type given by the second structure theorem, [3, p. 179].

Before tackling the task at hand we give some preliminary definitions and results.

Let J be a Jordan ring. If a is an element in J then U_a is the linear mapping on J given by $U_a(b) = 2a \cdot (a \cdot b) - a^{\cdot 2} \cdot b$. Defining primeness as did Chester Tsai in [5], J is *prime* provided that if A and B are ideals in J such that $U_A(B) = 0$, then $A = 0$ or $B = 0$. Clearly if A and B are ideals in J and $A \cdot B = 0$ then $U_A(B) = 0$. Q contained in J is said to be a *quadratic* *ideal* of J provided Q is an additive subgroup of $(J, +)$ and $U_Q(J) \subseteq Q$. A nonempty set $\{Q_i\}$ of nonzero quadratic ideals will be called a *direct system* provided $Q_i \cap Q_j = 0$ for $i \neq j$ and that if $\{Q_n : n \in N\}$ and $(Q_l : l \in L\}$ are subsets of $\{Q_i\}$, where $N \cap L = \emptyset$, then the quadratic ideal generated by ΣQ_n intersected with the quadratic ideal generated by ΣQ_l is 0. A direct system $\{Q_i\}$ is infinite if there are infinitely many elements in the set $\{Q_i\}$. One can easily see that this is an analogue of direct sums of ideals of an associative ring. An element a of J is said to be *regular* if U_a is injective. J is said to be a *Jordan integral domain* if $J \neq 0$ and every nonzero element in J is regular. If J is a Jordan integral domain then J is said to satisfy the common multiple property, cmp, if given any nonzero elements a, b of J then

$$U_a(J) \cap U_b(J) \neq 0.$$

Definition. The Jordan ring J' is said to be a *Jordan ring of quotients* for J if:

(1) there exists an isomorphism $f: J \to J'$;

(2) every regular element of J is invertible in J'; and

(3) every element of J' is of the form $U_{f(a)}^{-1}(f(b))$ for a, b in J with a regular in J.

Let R be an associative ring with involution $*$; let $H(R) = \{x \in R: x^* = x\}$; and let $K(R) = \{x \in R: x^* = -x\}$. An ideal A of R is said to be a *$*$-ideal* if $A = A^*$. R is said to be *$*$-prime* if for all nonzero$*$-ideals A, B of R, $AB \neq 0$. R is said to be *$*$-simple* if $R^2 \neq 0$ and it contains no proper nonzero $*$-ideals. We shall say a ring has characteristic $\neq 2$ if $2x = 0$ implies $x = 0$. We shall consider $H(R)$ as a subring of R^+ with the product $a \cdot b = ab + ba$ where the product on the right side is the product in the associative ring R, so that $U_a(b) = 4aba$.

General Assumption. Unless otherwise stated we are assuming R is an associative ring of characteristic $\neq 2$ with involution $*$ and $H = H(R)$ is a prime Jordan ring.

Let $M = \Sigma\{A: A$ is an ideal of R contained in $K(R)\}$. Thus, M is an ideal of R contained in $K(R)$ so that $M^3 = 0$. We shall use T to denote R/M. One can easily see that T is a $*$-prime associative ring of characteristic $\neq 2$ with involution $*$ given by $(x + M)^* = x^* + M$ and that the canonical homomorphism from R onto T establishes an isomorphism between H and $H(T)$.[1] The $*$-primeness of T is equivalent to T containing a prime ideal P such that $P \cap P^* = 0$, Martindale [4].

[1] Since we are taking rings of quotients we may assume that R is an algebra over a commutative associative ring with 1/2.

Erickson and Montgomery [1] have proven that our general assumption implies that H is nondegenerate in the sense that $U_a(H) = 0$ for $a \in H$ implies $a = 0$.

LEMMA A. *Let A be a nonzero left ideal of T.*

(i) *If $C = A_L(A) = \{x \in T: xA = 0\}$ then $C \cap C^* = 0$.*

(ii) *If $A \cap H(T) = 0$ then $A^*A \subseteq A \cap K(T)$.*

(iii) *If $A \cap H(T) = 0$ then $Q = \{a + a^*: a \in A\}$ is a quadratic ideal of $H(T)$.*

Proof. (i) Since C is the left annihilator of a left ideal, C is an ideal so that $C \cap C^*$ is a^*-ideal. $(C \cap C^*)A = 0$ implies $C \cap C^* \subseteq P$ or $A \subseteq P$ and $C \cap C^* \subseteq P^*$ or $A \subseteq P^*$. But $A \neq 0$ implies $C \cap C^*$ is contained in one of P or P^* and hence in both. Thus, $C \cap C^* = 0$.

(ii) $A \cap H(T) = 0$ implies $a^*b + b^*a = 0$ for all a, b in A. Thus, $A^*A \subseteq A \cap K(T)$.

(iii) Let $h \in H(T)$ and $a \in A$, so that $a^*ha \in A \cap H(T) = 0$ and by nondegeneracy $aha^* = 0$. Thus, $U_{a+a^*}(h) \in Q$ for all h in $H(T)$, and Q is a quadratic ideal.

LEMMA B. *If $h \neq 0$ is a regular element in the Jordan ring H then $\bar{h} = h + M$ is a regular element in the associative ring T in the sense that if $\bar{x} = x + M$ is in T and $\overline{xh} = 0$ then $x \in M$.*

Proof. Suppose $h \neq 0$ is regular in H, and $A_L(h) = \{x \in R: xh \in M\}$. It suffices to show $A_L(h) \subseteq K(R)$. Let $x \in A_L(h)$. $U_h(x + x^*) = 0$ so that by the regularity of h we have $x + x^* = 0$, and, hence, $A_L(h) \subseteq K(R)$.

COROLLARY. *If x is a regular element of $H(T)$ and if y is an element of T such that $xy = 0$ then $y = 0$.*

LEMMA C. *If ΣA_i is a direct sum of left ideals in T and $A_i \cap H(T) \neq 0$ for all i then $\{Q_i = A_i \cap H(T)\}$ is a direct system of quadratic ideals in $H(T)$.*

Proof. Let $\{Q_n = A_n \cap H(T): n \in N\}$ and $\{Q_l = A_l \cap H(T): l \in L\}$, where $N \cap L = \emptyset$. Let $X = (\Sigma_n A_n) \cap H(T)$ and $Y = (\Sigma_L A_l) \cap H(T)$. Clearly X, Y and the Q_i's are quadratic ideals in $H(T)$. $X \cap Y = 0$ since

$$X \cap Y \subseteq \Sigma_N A_n \cap \Sigma_L A_l = 0.$$

Thus, $\{Q_i\}$ is a direct system.

LEMMA D. *If ΣA_i is an infinite direct sum of nonzero left ideals in T such* *that $A_i{}^*A_i = 0 = A_i \cap H(T)$ for all i then $H(T)$ contains an infinite direct system of quadratic ideals.*

Proof. Let $Q_i = \{a + a^*: a \in A_i\}$. $Q_i = 0$ implies $A_i \subseteq K(T)$ which in turn implies $A_i = 0$ in T. Thus, $Q_i \neq 0$ for all i.

By the $*$-primeness of T and the fact that $A_i{}^*A_i = 0$, 0, we see that infinitely many of the A_i's are in P or infinitely many are in P^*. Thus, we may assume $A_i \subseteq P$ for all i. Let $\{Q_n : n \in N\}$ and $\{Q_l : l \in L\}$ be finite subsets of $\{Q_i\}$ where $N \cap L = \emptyset$. Let $X = \Sigma Q_n = \{a + a^*: a \in \Sigma_N A_n\}$ and $Y = \Sigma Q_l = \{a + a^*: a \in \Sigma_L A_l\}$. Since $\Sigma_N A_n \subseteq P$, $\Sigma_L A_l \subseteq P$ and $P \cap P^* = 0$, we see that X and Y are quadratic ideals, by Lemma A. Let $x \in X \cap Y$, so that for some choice of a_n's and a_l's, $x = \Sigma_N(a_n + a_n{}^*) = \Sigma_L(a_l + a_l{}^*)$. Thus, $x = \Sigma_N a_n + p^*$ and $x = \Sigma_L a_l + p_1{}^*$, where $p_1{}^*$, $p^* \in P^*$, so that $\Sigma_N a_n - \Sigma_L a_l = p_1{}^* - p^* = 0$, since the right side is in P^* and the left side is in P. This gives us that $\Sigma_N a_n \in \Sigma_N A_n \cap \Sigma_L A_l = 0$ so that $x = 0$. Hence, $X \cap Y = 0$ and $H(T)$ contains an infinite direct system.

LEMMA E. *If $H(T)$ satisfies either ACC or DCC on quadratic ideals then $H(T)$ contains no infinite direct system of quadratic ideals.*

Proof. Suppose $\{Q_i\}$ is an infinite direct system of quadratic ideals. Let X_n be the quadratic ideal generated by $\Sigma_1{}^n Q_i$ so that $X_1 \subset X_2 \subset X_3 \subset \cdots$ is a properly ascending chain of quadratic ideals. Let Y_n be the quadratic ideal generated by $\Sigma_n{}^\infty Q_i$. We claim that if $m < n$ then $Q_m \cap Y_n = 0$. One can see this by expressing Y_n as the union of Z_k's $k \geqslant n$, where Z_k is the quadratic ideal generated by $\Sigma_n{}^k Q_i$ and noticing $Z_k \cap Q_m = 0$ for all k.

THEOREM 1. *If $H(T)$ contains no infinite system of quadratic ideals then T contains no infinite direct sum of left ideals.*

Proof. Suppose ΣA_i is an infinite direct sum of left ideals of T indexed by the positive integers. By Lemma C, we may assume $A_i \cap H(T) = 0$ for all i. By Lemma D, we may assume $A_i{}^*A_i \neq 0$ for all i. By Lemma A, $A_i{}^*A_i \subseteq K(T)$. Thus, each A_i contains a nonzero skew, say a_i. Suppose that for each n, $(Ta_{2n} + Ta_{2n+1}) \cap H(T) \neq 0$. By Lemma C, we would have an infinite direct system since $\Sigma(Ta_{2n} + Ta_{2n+1})$ is an infinite direct sum. Thus, for some choice of n, $(Ta_{2n} + Ta_{2n+1}) \cap H(T) = 0$. We shall show that this cannot occur and then our proof will be complete.

$$a_{2n+1}ta_{2n} + a_{2n}t^*a_{2n+1} = 0$$

for all $t \in T$. Thus, $a_{2n+1}Ta_{2n} = a_{2n}Ta_{2n+1} = 0$. But this contradicts Lemma A.

Let A be the left annihilator of $S \subseteq T$ and B be the right annihilator of A

so that $A = A_L(S) = \{x \in T: xS = 0\}$ and $B = \{x \in T: Ax = 0\}$. It is easy to see $A = A_L(B)$. Therefore, whenever we have a properly ascending chain of left annihilators we have a corresponding properly descending chain of right annihilators.

Lemma F. *Let $A = A_L(S)$ and $B = A_R(A)$ both be nonzero and H be a Jordan integral domain. Then*

(i) $A \cap K(T) = 0$ *if and only if* $B \cap K(T) = 0$;
(ii) *if* $A \cap K(T) \neq 0$ *then* $B = A^*$.

Proof. (i) Suppose $A \cap K(T) \neq 0$ and $B \cap K(T) = 0$. By the corollary, $A \cap H(T) = 0 = B \cap H(T)$. Let $0 \neq k \in A \cap K(T)$ and $b \in B$.

$$k^2 \in A \cap H(T) = 0$$

so that $(bk - kb^*)k = -kb^*k \in A \cap H(T) = 0$. Therefore, $bk - kb^* = 0$, by the corollary, and $Bk \subseteq B \cap K(T) = 0$, contrary to Lemma A. By symmetry (i) is proven.

(ii) If $A \cap K(T) \neq 0$ then $B \cap K(T) \neq 0$, by (i). Let $0 \neq k \in A \cap K(T)$ and $0 \neq k_1 \in B \cap K(T)$. As shown above $bk - kb^* = 0$. Thus,

$$\begin{aligned} kB^*Bk &= \{\Sigma kb_j{}^*b_nk: b_j, b_n \in B\} \\ &= \{\Sigma b_jkkb_n{}^*: b_j, b_n \in B\} = 0 \end{aligned}$$

since $k^2 = 0$. Therefore, $B^*B = 0$ by Lemma A, so that $B^* \subseteq A$. Similarly $AA^* = 0$ so that $A^* \subseteq B$.

Theorem 2. *If H is a Jordan integral domain and contains no infinite direct system of quadratic ideals then T satisfies the Goldie conditions.*

Proof. Since H is isomorphic to $H(T)$, it suffices to show that T satisfies ACC on left annihilators. Let $A_1 \subseteq A_2 \subseteq \cdots$ be an ascending chain of left annihilators and $B_1 \supseteq B_2 \supseteq \cdots$ be the corresponding descending chain of right annihilators. By Lemma F we may assume $A_i \cap K(T) = 0 = B_i \cap K(T)$ for all i. By Lemma A, $A_i{}^*A_i = 0 = B_iB_i{}^*$. Thus, $B_i \subseteq P$ or $B_i \subseteq P^*$ for all i, and, hence, $A_i = 0$, P, or P^*, for all i.

If S is an associative ring with involution $*$ and W is a ring of quotients for S then $*$ given by $(a^{-1}b)^* = b^*a^{*-1}$ is an involution on W where $a, b \in S$ and a is not a divisor of 0.

The following theorem takes care of the integral domain situation as described by Jacobson for our restricted case.

Theorem 3. *If H is a Jordan integral domain then the following are equivalent*:

(i) *H satisfies the* cmp;

(ii) *H contains no infinite direct system of quadratic ideals*;

(iii) *T has a ring of quotients with involution which is one of the following; a divison ring D, $D \oplus D^0$ where D is a division ring and D^0 is an antiisomorphic copy of D with the exchange involution, or 2×2 matrices over a field whose set of symmetric elements is equal to its center.*

Proof. (i) implies (ii) is clear.

(ii) implies (iii).

By Theorem 2 T has a ring of quotients, say W. It is easy to see that W is $*$-simple. By Lemma B, every nonzero element in $H(T)$ is invertible in W. Let $a^{-1}b$ be an arbitrary element in $H(W)$. We may assume $a \in H(T)$. $a^{-1}b = b^*a^{-1}$ so that $ba = ab^*$. ba is invertible in $H(W)$. Thus, $a^{-1}b$ is invertible in $H(W)$. Therefore, W is an associative $*$-simple ring in which every symmetric element is invertible. J. Marshall Osborn classified all such rings as those given in (iii) (see [3, p. 166]).

(iii) implies (i) is easy to check.

LEMMA G. *Let $A_1 \subset A_2 \subset \cdots$ be a properly ascending chain of left annihilators and $B_1 \supset B_2 \supset \cdots$ be the corresponding properly descending chain of right annihilators*

(i) *If $H(T)$ satisfies ACC on quadratic ideals then $A_i \cap H(T) \neq 0$ for some i.*

(ii) *If $H(T)$ satisfies DCC on quadratic ideals then $B_i \cap H(T) \neq 0$ for all i.*

Proof. Suppose $A_i \cap H(T) = 0$ for all i. Let $Q_i = \{a + a^*: a \in A_i\}$. By Lemma A, Q_i is a quadratic ideal. There exists an m such that $Q_m = Q_{m+i}$ for $i \geqslant 0$.

Let $x \in A_{m+i}$ for $i \geqslant 1$, so that $x = y + y^* - x^*$ for some y in A_m. $y^* - x^* \in A^*_{m+i}$ implies $A_{m+i} \subseteq A_m + A^*_{m+i}$ so that

$$B^*_{m+i}(A_{m+i}B_m) \subseteq (B_{m+i}A_{i+m})\, B_m = 0.$$

Thus, $B_{m+i} \subseteq P^*$ and $A_{m+i}B_m \subseteq P^*$ or $B_{m+i} \subseteq P$ and $A_{m+i}B_m \subseteq P$. Thus, $P \neq 0$. Assume $B_{m+i} \subseteq P$ so that $P^* \subseteq A_{m+i}$. Now let p be a nonzero element in P. $a^*p^* + pa \in A_{m+i} \cap H(T) = 0$ for all A_{m+i}. Thus, $a^*p^* = -pa$ is in $P \cap P^* = 0$ so that $PA_{m+i} = 0$ and $A_{m+i} \subseteq P^*$ for $i \geqslant 1$. Thus, $A_{m+i} = P^*$ for $i \geqslant 1$, contrary to assumption.

(ii) It suffices to show that there exists an i such that $B_i \cap H(T) \neq 0$. Suppose $B_i \cap H(T) = 0$ for all i. Let $Q_i = \{b + b^*: b \in B_i\}$. There exists an m such that $Q_m = Q_{m+i}$ for $i \geqslant 0$.

Now $B_i \subseteq B_{m+i} + B_m{}^*$ for $i \geqslant 1$ so that $(A_{m+i}B_m)\, A_m{}^* = 0$, and we see that $A_m \subseteq P$ and $A_{m+i}B_m \subseteq P$ or $A_m \subseteq P^*$ and $A_{m+i}B_m \subseteq P^*$. Continue in a manner similar to that of (i).

THEOREM 4. *If $H(T)$ satisfies ACC or DCC on quadratic ideals then T satisfies ACC on left annihilator ideals.*

Proof. Suppose T does not satisfy ACC on left annihilator ideals. Let $A_1 \subset A_2 \subset \cdots$ be a properly ascending chain of left annihilator ideals and $B_1 \supset B_2 \supset \cdots$ be the corresponding properly descending chain of right annihilator ideals.

Suppose $H(T)$ satisfies ACC on quadratic ideals. There exists an m such that $A_m \cap H(T) = A_{m+i} \cap H(T)$ for all $i \geqslant 0$. Using Lemma G, let h be a nonzero element in $A_m \cap H(T) \subseteq A_{m+i}$ for all $i \geqslant 0$. Let $y \in A_{m+i}i \geqslant 0$, so that $hy + y^*h$ is in $A_{m+i} \cap H(T) = A_m \cap H(T)$. Thus, $0 = (hy + y^*h)B_m = hyB_m$, since $h \in A_m$. We see that $h(A_{m+i}B_m) = 0$ for $i \geqslant 0$. But if $i \geqslant 1$ then $A_{m+i}B_m \neq 0$ and we have a contradiction to Lemma A.

Suppose $H(T)$ satisfies DCC on quadratic ideals. By Lemma G, $B_i \cap H(T) \neq 0$ for all i. There exists an m such that $B_m \cap H(T) = B_{m+i} \cap H(T)$ for all $i \geqslant 0$. Let h be a nonzero element in $B_m \cap H(T)$ and let y be an arbitrary element in B_m so that $yh + hy^* \in B_m \cap H(T) = B_{m+i} \cap H(T)$ for all $i \geqslant 0$. Thus, $0 = A_{m+i}(yh + hy^*) = A_{m+i}yh$. Therefore, $(A_{m+i}B_m)h = 0$ for $i \geqslant 0$, contrary to Lemma A.

THEOREM 5. *If S is a $*$-prime Goldie ring then the ring of quotients is a $*$-simple Artinian ring W.*

Proof. By Goldie's theorem for semi-prime rings, W is semi-simple Artinian, so that $W \simeq \oplus\Sigma_1{}^m W_i$ such that each W_i is a simple Artinian ring. Let A be a $*$-ideal of W. $A \simeq \Sigma_1{}^m(W_1 \cap A')$ where A' is the isomorphic image of A in $\Sigma_1{}^m W_i$. By simplicity of W_i, $W_i \cap A' = W_i$ or 0.

We want to show $A' \cap W_i = W_i$ for $i = 1,..., m$. If $W_i \cap A' = 0$ for some i, then $B = \{x \in S \colon x(A \cap S) = 0\}$ is a nonzero ideal such that $B(A \cap S) = 0$. But $A \cap S$ is a nonzero $*$-ideal so that $B \subseteq P \cap P^* = 0$.

Theorem 5 completes the proof of our main theorem.

THEOREM 6. *If H satisfies either ACC or DCC or quadratic ideals and W is the ring of quotients for T, then $H(W)$ is a Jordan ring of quotients for H.*

Proof. We embed H in $H(W)$ by $H \simeq_u H(T) \to_g H(W)$ where g is the isomorphism from T into W. Call this composition f. By Lemma B, all that remains to be shown is that every element in $H(W)$ has the form $U^{-1}_{(f)a}(f(b))$ for some $a, b \in H$ with a regular in H.

Let $g(x)^{-1}g(y)$ be an arbitrary element in $H(W)$. We may assume $x^* = x$. $g(x)^{-1}g(y)$ in $H(W)$ implies that $g(x)^{-1}g(y) = g(y^*)\, g(x)^{-1}$ so that $xy^* = yx$. Thus, yx is in $H(T)$ so that there exist elements a, b in H such that $u(a) = 2x$ and $u(b) = yx$.

$$\begin{aligned} U_{f(a)}^{-1}(f(b)) &= 4f(a)^{-1}f(b)\, f(a)^{-1} \\ &= g(x)^{-1}g(yx)\, g(x)^{-1} \\ &= g(x)^{-1}g(y). \end{aligned}$$

COROLLARY. *Let J be a Jordan algebra with characteristic $\neq 2$ which is generated by 1 and two elements or which is special and is generated by 1 and three elements. Then if J is prime and satisfies ACC on quadratic ideals then J has a Jordan ring of quotients which is isomorphic to the Jordan algebra of symmetric elements of a *-simple Artinian ring.*

This Corollary follows from [3, Corollaries 1 and 2, p. 77] and the main theorem.

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